8.6 Linearization of Nonlinear Systems

In this section we show how to perform linearization of systems described by nonlinear differential equations. The procedure introduced is based on the Taylor series expansion and on knowledge of nominal system trajectories and nominal system inputs.

We will start with a simple scalar first-order nonlinear dynamic system

$$\dot{x}(t) = \mathcal{F}(x(t), f(t)), \quad x(t_0) \quad \text{given}$$

Assume that under usual working circumstances this system operates along the trajectory $x_n(t)$ while it is driven by the system input $f_n(t)$. We call $x_n(t)$ and $f_n(t)$, respectively, the *nominal system trajectory* and the *nominal system input*.

On the nominal trajectory the following differential equation is satisfied

$$\dot{x}_n(t) = \mathcal{F}(x_n(t), u_n(t))$$

Assume that the motion of the nonlinear system is in the neighborhood of the nominal system trajectory, that is

$$x(t) = x_n(t) + \Delta x(t)$$

where $\Delta x(t)$ represents a small quantity. It is natural to assume that the system motion in close proximity to the nominal trajectory will be sustained by a system input which is obtained by adding a small quantity to the nominal system input

$$f(t) = f_n(t) + \Delta f(t)$$

For the system motion in close proximity to the nominal trajectory, we have

$$\dot{x}_n(t) + \Delta\dot{x}(t) = \mathcal{F}(x_n(t) + \Delta x(t), f_n(t) + \Delta f(t))$$

Since $\Delta x(t)$ and $\Delta f(t)$ are small quantities, the right-hand side can be expanded into a Taylor series about the nominal system trajectory and input, which produces

$$\dot{x}_n(t) + \Delta\dot{x}(t) = \mathcal{F}(x_n, f_n) + \frac{\partial \mathcal{F}}{\partial x}(x_n, f_n)\Delta x(t) + \frac{\partial \mathcal{F}}{\partial u}(x_n, f_n)\Delta u(t)$$
$$+ \text{ higher–order terms}$$

Canceling higher-order terms (which contain very small quantities $\Delta x^2, \Delta f^2, \Delta x \Delta f, \Delta x^3, \ldots$), the linear differential equation is obtained

$$\Delta\dot{x}(t) = \frac{\partial \mathcal{F}}{\partial x}(x_n, f_n)\Delta x(t) + \frac{\partial \mathcal{F}}{\partial f}(x_n, f_n)\Delta f(t)$$

The *partial derivatives in the linearization procedure are evaluated at the nominal points*. Introducing the notation

$$a_0(t) = -\frac{\partial \mathcal{F}}{\partial x}(x_n, f_n), \qquad b_0 = \frac{\partial \mathcal{F}}{\partial f}(x_n, f_n)$$

the linearized system can be represented as

$$\Delta \dot{x}(t) + a_0(t)\Delta x(t) = b_0(t)\Delta f(t)$$

In general, the obtained linear system is time varying. Since in this course we study only time invariant systems, we will consider only those examples for which the linearization procedure produces time invariant systems. It remains to find the initial condition for the linearized system, which can be obtained from

$$\Delta x(t_0) = x(t_0) - x_n(t_0)$$

Similarly, we can linearize the second-order nonlinear dynamic system

$$\ddot{x} = \mathcal{F}\Big(x, \dot{x}, f, \dot{f}\Big), \qquad x(t_0), \;\; \dot{x}(t_0) \;\; \textbf{given}$$

by assuming that

$$x(t) = x_n(t) + \Delta x(t), \;\; \dot{x}(t) = \dot{x}_n(t) + \Delta \dot{x}(t)$$

$$f(t) = f_n(t) + \Delta f(t), \;\; \dot{f}(t) = \dot{f}_n(t) + \Delta \dot{f}(t)$$

and expanding

$$\ddot{x}_n + \Delta \ddot{x} = f\Big(x_n + \Delta x_n, \dot{x}_n + \Delta \dot{x}, f_n + \Delta f, \dot{f}_n + \Delta \dot{f}\Big)$$

into a Taylor series about nominal points $x_n, \dot{x}_n, f_n, \dot{f}_n$, which leads to

$$\Delta \ddot{x}(t) + a_1 \Delta \dot{x}(t) + a_0 \Delta x(t) = b_1 \Delta \dot{f}(t) + b_0 \Delta f(t)$$

where the corresponding coefficients are evaluated at the nominal points as

$$a_1 = -\frac{\partial \mathcal{F}}{\partial \dot{x}}\Big(x_n, \dot{x}_n, f_n, \dot{f}_n\Big), \quad a_0 = -\frac{\partial \mathcal{F}}{\partial x}\Big(x_n, \dot{x}_n, f_n, \dot{f}_n\Big)$$

$$b_1 = \frac{\partial \mathcal{F}}{\partial \dot{f}}\Big(x_n, \dot{x}_n, f_n, \dot{f}_n\Big), \quad b_0 = \frac{\partial \mathcal{F}}{\partial f}\Big(x_n, \dot{x}_n, f_n, \dot{f}_n\Big)$$

The initial conditions for the second-order linearized system are obtained from

$$\Delta x(t_0) = x(t_0) - x_n(t_0), \quad \Delta \dot{x}(t_0) = \dot{x}(t_0) - \dot{x}_n(t_0)$$

Example 8.15: The mathematical model of a stick-balancing problem is

$$\ddot{\theta}(t) = \sin\theta(t) - f(t)\cos\theta(t) = \mathcal{F}(\theta(t), f(t))$$

where $f(t)$ is the horizontal force of a finger and $\theta(t)$ represents the stick's angular displacement from the vertical.

This second-order dynamic system is linearized at the nominal points $\dot{\theta}_n(t) = \theta_n(t) = 0,\ f_n(t) = 0$, producing

$$a_1 = -\frac{\partial \mathcal{F}}{\partial \dot{\theta}} = 0, \quad a_0 = -\left(\frac{\partial \mathcal{F}}{\partial \theta}\right)_{\big|_n} = -(\cos\theta + f\sin\theta)_{\big|_{\substack{\theta_n(t)=0 \\ f_n(t)=0}}} = -1$$

$$b_1 = \frac{\partial \mathcal{F}}{\partial \dot{f}} = 0, \quad b_0 = \left(\frac{\partial \mathcal{F}}{\partial f}\right)_{\big|_n} = -(\cos\theta)_{\big|_{\theta_n(t)=0}} = -1$$

The linearized equation is given by

$$\ddot{\theta}(t) - \theta(t) = -f(t)$$

Note that $\Delta\theta(t) = \theta(t)$, $\Delta f(t) = f(t)$ since $\theta_n(t) = 0$, $f_n(t) = 0$. It is important to point out that the same linearized model could have been obtained by setting $\sin\theta(t) \approx \theta(t)$, $\cos\theta(t) \approx 1$, which is valid for small values of $\theta(t)$.

We can extend the presented linearization procedure to an n-order nonlinear dynamic system with one input and one output in a straightforward way. However, for multi-input multi-output systems this procedure becomes cumbersome. Using the state space model, the linearization procedure for the multi-input multi-output case is simplified.

Consider now the general nonlinear dynamic control system in matrix form

$$\frac{d}{dt}\mathrm{x}(t) = \mathcal{F}(\mathrm{x}(t), \mathbf{f}(t)), \quad \mathrm{x}(t_0) \;\; \textbf{given}$$

where $\mathrm{x}(t)$, $\mathbf{f}(t)$, and $\mathcal{F}$ are, respectively, the n-dimensional system state space vector, the r-dimensional input vector, and the n-dimensional vector function. Assume that the nominal (operating) system trajectory $\mathrm{x}_n(t)$ is known and that the nominal system input that keeps the system on the nominal trajectory is given by $\mathbf{f}_n(t)$.

Using the same logic as for the scalar case, we can assume that the actual system dynamics in the immediate proximity of the system nominal trajectories can be approximated by the first terms of the Taylor series. That is, starting with

$$\mathbf{x}(t) = \mathbf{x}_n(t) + \Delta\mathbf{x}(t), \qquad \mathbf{f}(t) = \mathbf{f}_n(t) + \Delta\mathbf{f}(t)$$

and

$$\frac{d}{dt}\mathbf{x}_n(t) = \mathcal{F}(\mathbf{x}_n(t), \mathbf{f}_n(t))$$

we expand the right-hand side into the Taylor series as follows

$$\frac{d}{dt}\mathbf{x}_n + \frac{d}{dt}\Delta\mathbf{x} = \mathcal{F}(\mathbf{x}_n + \Delta\mathbf{x}, \mathbf{f}_n + \Delta\mathbf{f})$$

$$= \mathcal{F}(\mathbf{x}_n, \mathbf{f}_n) + \left(\frac{\partial \mathcal{F}}{\partial \mathbf{x}}\right)_{\big|\substack{\mathbf{x}_n(t) \\ \mathbf{f}_n(t)}} \Delta\mathbf{x} + \left(\frac{\partial \mathcal{F}}{\partial \mathbf{f}}\right)_{\big|\substack{\mathbf{x}_n(t) \\ \mathbf{f}_n(t)}} \Delta\,\mathbf{f} + \text{higher–order terms}$$

Higher-order terms contain at least quadratic quantities of $\boldsymbol{\Delta}\mathrm{x}$ and $\boldsymbol{\Delta}\mathbf{f}$. Since $\boldsymbol{\Delta}\mathrm{x}$ and $\boldsymbol{\Delta}\mathbf{f}$ are small their squares are even smaller, and hence the high-order terms can be neglected. Neglecting higher-order terms, an approximation is obtained

$$\frac{d}{dt}\boldsymbol{\Delta}\mathrm{x}(t) = \left(\frac{\partial \mathcal{F}}{\partial \mathrm{x}}\right)_{|_{\mathrm{f}_n(t)}^{\mathrm{x}_n(t)}} \boldsymbol{\Delta}\mathrm{x}(t) + \left(\frac{\partial \mathcal{F}}{\partial \mathbf{f}}\right)_{|_{\mathrm{f}_n(t)}^{\mathrm{x}_n(t)}} \boldsymbol{\Delta}\mathbf{f}(t)$$

The partial derivatives represent the Jacobian matrices given by

$$\left(\frac{\partial \mathcal{F}}{\partial \mathrm{x}}\right)_{|_{\mathrm{f}_n(t)}^{\mathrm{x}_n(t)}} = \mathbf{A}^{n\times n} = \begin{bmatrix} \frac{\partial \mathcal{F}_1}{\partial x_1} & \frac{\partial \mathcal{F}_1}{\partial x_2} & \cdots & \cdots & \frac{\partial \mathcal{F}_1}{\partial x_n} \\ \frac{\partial \mathcal{F}_2}{\partial x_1} & \cdots & \cdots & \cdots & \frac{\partial \mathcal{F}_2}{\partial x_n} \\ \cdots & \cdots & \frac{\partial \mathcal{F}_i}{\partial x_j} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \mathcal{F}_n}{\partial x_1} & \frac{\partial \mathcal{F}_n}{\partial x_2} & \cdots & \cdots & \frac{\partial \mathcal{F}_n}{\partial x_n} \end{bmatrix}_{|_{\mathrm{f}_n(t)}^{\mathrm{x}_n(t)}}$$

$$\left(\frac{\partial \mathcal{F}}{\partial \mathrm{f}}\right)_{|_{\mathrm{f}_n(t)}^{\mathrm{x}_n(t)}} = \mathrm{B}^{n\times r} = \begin{bmatrix} \frac{\partial \mathcal{F}_1}{\partial f_1} & \frac{\partial \mathcal{F}_1}{\partial f_2} & \cdots & \cdots & \frac{\partial \mathcal{F}_1}{\partial f_r} \\ \frac{\partial \mathcal{F}_2}{\partial f_1} & \cdots & \cdots & \cdots & \frac{\partial \mathcal{F}_2}{\partial f_r} \\ \cdots & \cdots & \frac{\partial \mathcal{F}_i}{\partial f_j} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \mathcal{F}_n}{\partial f_1} & \frac{\partial \mathcal{F}_n}{\partial f_2} & \cdots & \cdots & \frac{\partial \mathcal{F}_n}{\partial f_r} \end{bmatrix}_{|_{\mathrm{f}_n(t)}^{\mathrm{x}_n(t)}}$$

Note that the Jacobian matrices have to be evaluated at the nominal points, that is, at $\mathrm{x}_n(t)$ and $\mathrm{f}_n(t)$. With this notation, the linearized system has the form

$$\frac{d}{dt}\Delta\mathrm{x}(t) = \mathrm{A}\Delta\mathrm{x}(t) + \mathrm{B}\Delta\mathrm{u}(t), \quad \Delta\mathrm{x}(t_0) = \mathrm{x}(t_0) - \mathrm{x}_n(t_0)$$

The output of a nonlinear system satisfies a nonlinear algebraic equation, that is

$$\mathrm{y}(t) = \mathcal{G}(\mathrm{x}(t), \mathrm{f}(t))$$

This equation can also be linearized by expanding its right-hand side into a Taylor series about nominal points $\mathrm{x}_n(t)$ and $\mathbf{f}_n(t)$. This leads to

$$\mathbf{y}_n + \Delta\mathbf{y} = \mathcal{G}(\mathrm{x}_n, \mathbf{f}_n) + \left(\frac{\partial \mathcal{G}}{\partial \mathrm{x}}\right)_{\big|\substack{\mathrm{x}_n(t)\\ \mathbf{f}_n(t)}} \Delta\mathrm{x} + \left(\frac{\partial \mathcal{G}}{\partial \mathbf{f}}\right)_{\big|\substack{\mathrm{x}_n(t)\\ \mathbf{f}_n(t)}} \Delta\mathbf{f}$$
$$+ \text{ higher–order terms}$$

Note that $\mathbf{y}_n$ cancels term $\mathcal{G}(\mathrm{x}_n, \mathbf{f}_n)$. By neglecting higher-order terms, the linearized part of the output equation is given by

$$\Delta\mathbf{y}(t) = \mathbf{C}\Delta\mathrm{x}(t) + \mathbf{D}\Delta\mathbf{f}(t)$$

where the Jacobian matrices $\mathbf{C}$ and $\mathbf{D}$ satisfy

$$
\mathbf{C}^{p\times n} = \left(\frac{\partial \mathcal{G}}{\partial \mathbf{x}}\right)_{\Big|\substack{\mathbf{x}_n(t)\\ \mathbf{f}_n(t)}} = \begin{bmatrix} \frac{\partial \mathcal{G}_1}{\partial x_1} & \frac{\partial \mathcal{G}_1}{\partial x_2} & \cdots & \cdots & \frac{\partial \mathcal{G}_1}{\partial x_n} \\ \frac{\partial \mathcal{G}_2}{\partial x_1} & \cdots & \cdots & \cdots & \frac{\partial \mathcal{G}_2}{\partial x_n} \\ \cdots & \cdots & \frac{\partial \mathcal{G}_i}{\partial x_j} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \mathcal{G}_p}{\partial x_1} & \frac{\partial \mathcal{G}_p}{\partial x_2} & \cdots & \cdots & \frac{\partial \mathcal{G}_p}{\partial x_n} \end{bmatrix}_{\Big|\substack{\mathbf{x}_n(t)\\ \mathbf{f}_n(t)}}
$$

$$
\mathbf{D}^{p\times r} = \left(\frac{\partial \mathcal{G}}{\partial \mathbf{f}}\right)_{\Big|\substack{\mathbf{x}_n(t)\\ \mathbf{f}_n(t)}} = \begin{bmatrix} \frac{\partial \mathcal{G}_1}{\partial f_1} & \frac{\partial \mathcal{G}_1}{\partial f_2} & \cdots & \cdots & \frac{\partial \mathcal{G}_1}{\partial f_r} \\ \frac{\partial \mathcal{G}_2}{\partial f_1} & \cdots & \cdots & \cdots & \frac{\partial \mathcal{G}_2}{\partial f_r} \\ \cdots & \cdots & \frac{\partial \mathcal{G}_i}{\partial f_j} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \mathcal{G}_p}{\partial f_1} & \frac{\partial \mathcal{G}_p}{\partial f_2} & \cdots & \cdots & \frac{\partial \mathcal{G}_p}{\partial f} \end{bmatrix}_{\Big|\substack{\mathbf{x}_n(t)\\ \mathbf{f}_n(t)}}
$$

Example 8.16: Let a nonlinear system be represented by

$$\frac{dx_1(t)}{dt} = x_1(t)\sin x_2(t) + x_2(t)f(t)$$
$$\frac{dx_2(t)}{dt} = x_1(t)e^{-x_2(t)} + f^2(t)$$
$$y(t) = 2x_1(t)x_2(t) + x_2^2(t)$$

Assume that the values for the system nominal trajectories and input are known and given by x_{1n}, x_{2n}, and f_n. The linearized state space equation of this nonlinear system is obtained as

$$\begin{bmatrix} \Delta\dot{x}_1(t) \\ \Delta\dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \sin x_{2n} & x_{1n}\cos x_{2n} + f_n \\ e^{-x_{2n}} & -x_{1n}e^{-x_{2n}} \end{bmatrix} \begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \end{bmatrix} + \begin{bmatrix} x_{2n} \\ 2f_n \end{bmatrix} \Delta f(t)$$
$$\Delta y(t) = \begin{bmatrix} 2x_{2n} & 2x_{1n} + 2x_{2n} \end{bmatrix} \begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \end{bmatrix}$$

Having obtained the solution of this linearized system under the given system input $\Delta f(t)$, the corresponding approximation of the nonlinear system trajectories is

$$\mathrm{x}_n(t) + \Delta\mathrm{x}(t) = \begin{bmatrix} x_{1n}(t) \\ x_{2n}(t) \end{bmatrix} + \begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \end{bmatrix}$$

Example 8.17: Consider the mathematical model of a single-link robotic manipulator with a flexible joint given by

$$I\ddot{\theta}_1(t) + mgl\sin\theta_1(t) + k(\theta_1(t) - \theta_2(t)) = 0$$

$$J\ddot{\theta}_2(t) - k(\theta_1(t) - \theta_2(t)) = f(t)$$

where $\theta_1(t), \theta_2(t)$ are angular positions, I, J are moments of inertia, m and l are, respectively, link mass and length, and k is the link spring constant. Introducing the change of variables as

$$x_1(t) = \theta_1(t),\ x_2(t) = \dot{\theta}_1(t),\ x_3(t) = \theta_2(t),\ x_4(t) = \dot{\theta}_2(t)$$

the manipulator's state space nonlinear model is given by

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = -\frac{mgl}{I}\sin x_1(t) - \frac{k}{I}(x_1(t) - x_3(t))$$

$$\dot{x}_3(t) = x_4(t)$$

$$\dot{x}_4(t) = \frac{k}{J}(x_1(t) - x_3(t)) + \frac{1}{J}f(t)$$

Take the nominal points as $(x_{1n}, x_{2n}, x_{3n}, x_{4n}, f_n)$, then the matrices $\mathbf{A}$ and $\mathbf{B}$ are

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k+mgl\cos x_{1n}}{I} & 0 & \frac{k}{I} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{J} & 0 & -\frac{k}{J} & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{J} \end{bmatrix}$$

Assuming that the output variable is equal to the link's angular position, that is $y(t) = x_1(t)$, the matrices $\mathbf{C}$ and $\mathbf{D}$ are given by

$$\mathbf{C} = [1 \quad 0 \quad 0 \quad 0], \quad \mathbf{D} = 0$$